

STRICTLY CONVEX WULFF SHAPES AND C^1 CONVEX INTEGRANDS

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ABSTRACT. In this paper, it is shown that a Wulff shape is strictly convex if and only if its convex integrand is of class C^1 . Moreover, applications of this result are given.

1. INTRODUCTION

Let $\gamma : S^n \rightarrow \mathbb{R}_+$ be a continuous function, where n is a positive integer, S^n is the unit sphere in \mathbb{R}^{n+1} and $\mathbb{R}_+ = \{a \in \mathbb{R} \mid a > 0\}$. For any $\theta \in S^n$, we set

$$\Gamma_{\gamma, \theta} = \{x \in \mathbb{R}^{n+1} \mid x \cdot \theta \leq \gamma(\theta)\},$$

where the dot in the center stands for the standard dot product of two vectors $x, \theta \in \mathbb{R}^{n+1}$. Then, the following set \mathcal{W}_γ is called the *Wulff shape* associated with γ (see Figure 1).

$$\mathcal{W}_\gamma = \bigcap_{\theta \in S^n} \Gamma_{\gamma, \theta}.$$

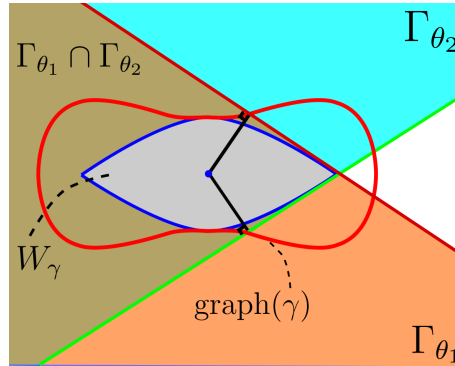


FIGURE 1. A Wulff shape \mathcal{W}_γ .

The Wulff shape \mathcal{W}_γ , which was firstly introduced by G. Wulff in [18], is known as a geometric model of a crystal at equilibrium (for instance, see [3, 13, 16, 17]). By definition, the Wulff shape \mathcal{W}_γ is compact, convex and it contains the origin of \mathbb{R}^{n+1} as an interior point; namely, \mathcal{W}_γ is a convex body such that the origin is contained in $\text{int}(\mathcal{W}_\gamma)$, where $\text{int}(\mathcal{W}_\gamma)$ stands for the set consisting of interior points

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of \mathcal{W}_γ (for details on convex bodies, see [15]). Conversely, it is known that any convex body containing the origin as an interior point is a Wulff shape associated with an appropriate support function ([16]). Thus, for any convex body $W \subset \mathbb{R}^{n+1}$ such that $\text{int}(W)$ contains the origin, there exists the non-empty set, denoted by $C_W^0(S^n, \mathbb{R}_+)$, consisting of continuous functions $\gamma : S^n \rightarrow \mathbb{R}_+$ such that $\mathcal{W}_\gamma = W$. Moreover, it is known that for any convex body W such that the origin is contained in $\text{int}(W)$, there exists the smallest element $\gamma_w \in C_W^0(S^n, \mathbb{R}_+)$ in the sense that $\gamma_w(\theta) \leq \gamma(\theta)$ is satisfied for any $\theta \in S^n$ and any $\gamma \in C_W^0(S^n, \mathbb{R}_+)$ ([16]). The function γ_w is called the *convex integrand* of W (for details on convex integrand, see Section 2).

Theorem 1. *Let $W \subset \mathbb{R}^{n+1}$ be a convex body containing the origin of \mathbb{R}^{n+1} as an interior point of W . Then, W is strictly convex if and only if its convex integrand γ_w is of class C^1 .*

A more restricted dual relationship than the one given in Theorem 1 has been obtained by F. Morgan as follows.

Theorem 2 ([7]). *Let $W \subset \mathbb{R}^{n+1}$ be a convex body containing the origin of \mathbb{R}^{n+1} as an interior point of W . Then, W is uniformly convex if and only if its convex integrand γ_w is of class $C^{1,1}$.*

For the definitions of uniform convexity and of class $C^{1,1}$, see [7]. As explained in p. 348 of [7], the notion of strict convexity (resp., class C^1) is certainly weaker than the one of uniform convexity (resp., class $C^{1,1}$) for Wulff shapes (resp., convex integrands). Moreover, “strict convexity” (resp., “class C^1 ”) is more common and easy to treat than “uniform convexity” (resp., “class $C^{1,1}$ ”). Thus, Theorem 1 may be regarded as a useful generalization of Theorem 2.

In Section 2, preliminaries are given. Proof of Theorem 1 is given in Section 3. In Section 4, applications of Theorem 1 are given.

2. PRELIMINARIES

2.1. Convex integrands. Let $\gamma : S^n \rightarrow \mathbb{R}_+$ be a continuous function. Set

$$\text{graph}(\gamma) = \{(\theta, \gamma(\theta)) \in \mathbb{R}^{n+1} - \{0\} \mid \theta \in S^n\},$$

where $(\theta, \gamma(\theta))$ is the polar plot expression for a point of $\mathbb{R}^{n+1} - \{0\}$. Let $\text{inv} : \mathbb{R}^{n+1} - \{0\} \rightarrow \mathbb{R}^{n+1} - \{0\}$ be the inversion with respect to the origin of \mathbb{R}^{n+1} , namely, $\text{inv} : \mathbb{R}^{n+1} - \{0\} \rightarrow \mathbb{R}^{n+1} - \{0\}$ is defined as follows:

$$\text{inv}(\theta, r) = \left(-\theta, \frac{1}{r}\right).$$

Let Γ_γ be the boundary of the convex hull of $\text{inv}(\text{graph}(\gamma))$ (see Figure 2).

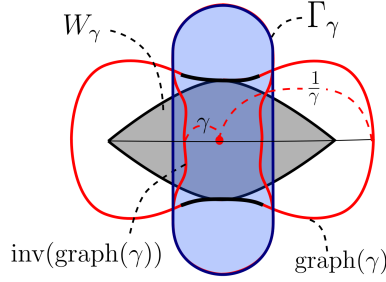
Definition 1 ([16]). A continuous function $\gamma : S^n \rightarrow \mathbb{R}_+$ satisfying $\Gamma_\gamma = \text{inv}(\text{graph}(\gamma))$ is called a *convex integrand*.

The following has been known.

Proposition 1 ([16, 12]). *For any $\gamma_1, \gamma_2 : S^n \rightarrow \mathbb{R}_+$, the following holds:*

$$\Gamma_{\gamma_1} = \Gamma_{\gamma_2} \Leftrightarrow \mathcal{W}_{\gamma_1} = \mathcal{W}_{\gamma_2}.$$

Proposition 1 implies the following:

FIGURE 2. A Wulff shape \mathcal{W}_γ and the convex hull Γ_γ .

Proposition 2. *Let W be a convex body in \mathbb{R}^{n+1} such that the origin of \mathbb{R}^{n+1} is contained in $\text{int}(W)$. Then, the following holds for any two $\gamma_1, \gamma_2 \in C_W^0(S^n, \mathbb{R}_+)$:*

$$\Gamma_{\gamma_1} = \Gamma_{\gamma_2}.$$

By Proposition 2, the following definition is well-defined:

Definition 2 ([16]). Let W be a convex body in \mathbb{R}^{n+1} such that the origin of \mathbb{R}^{n+1} is contained in $\text{int}(W)$. Define the unique function $\gamma_W : S^n \rightarrow \mathbb{R}_+$ as follows:

$$\text{graph}(\gamma_W) = \text{inv}(\Gamma_\gamma),$$

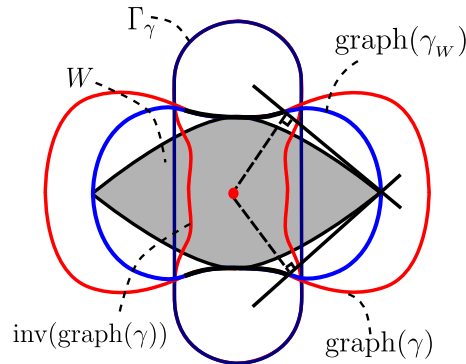
where γ is a function of $C_W^0(S^n, \mathbb{R}_+)$. The function γ_W is called the *convex integrand* of W .

By the construction of convex integrand of W and Proposition 2, the following holds:

Proposition 3 ([16]). *Let W be a convex body containing the origin as an interior point. Then, the following holds for any $\theta \in S^n$ and any $\gamma \in C_W^0(S^n, \mathbb{R}_+)$.*

$$\gamma_W(\theta) \leq \gamma(\theta).$$

Figure 3 illustrates Proposition 3.

FIGURE 3. $\gamma_W(\theta) \leq \gamma(\theta)$ for any $\theta \in S^n$.

2.2. Convex geometry in S^{n+1} . For any point $P \in S^{n+1}$, let $H(P)$ be the closed hemisphere centered at P ; namely, $H(P)$ is the set consisting of $Q \in S^{n+1}$ satisfying $P \cdot Q \geq 0$, where the dot in the center stands for the scalar product of two vectors $P, Q \in \mathbb{R}^{n+2}$.

Definition 3 ([12]). Let \widetilde{W} be a subset of S^{n+1} . Suppose that there exists a point $P \in S^{n+1}$ such that $\widetilde{W} \cap H(P) = \emptyset$. Then, \widetilde{W} is said to be *hemispherical*.

For any non-empty subset $\widetilde{W} \subset S^{n+1}$, the *spherical polar set of \widetilde{W}* , denoted by \widetilde{W}° , is defined as follows:

$$\widetilde{W}^\circ = \bigcap_{P \in \widetilde{W}} H(P).$$

Lemma 2.1 ([12]). *For any hemispherical finite subset $\widetilde{X} = \{P_1, \dots, P_k\} \subset S^{n+1}$, the following holds:*

$$\left\{ \frac{\sum_{i=1}^k t_i P_i}{\|\sum_{i=1}^k t_i P_i\|} \mid P_i \in \widetilde{X}, \sum_{i=1}^k t_i = 1, t_i \geq 0 \right\}^\circ = H(P_1) \cap \dots \cap H(P_k).$$

Lemma 2.1 is called *Maehara's lemma*.

Let P, Q be two points of S^{n+1} such that $(1-t)P + tQ$ is not the zero vector for any $t \in [0, 1]$. Then, the following arc is denoted by PQ :

$$PQ = \left\{ \frac{(1-t)P + tQ}{\|(1-t)P + tQ\|} \in S^{n+1} \mid 0 \leq t \leq 1 \right\}.$$

Definition 4 ([4]). Let $\widetilde{W} \subset S^{n+1}$ be a hemispherical subset.

- (1) Suppose that $PQ \subset \widetilde{W}$ for any $P, Q \in \widetilde{W}$. Then, \widetilde{W} is said to be *spherical convex*.
- (2) Suppose that \widetilde{W} is closed, spherical convex and has an interior point. Then, \widetilde{W} is said to be a *spherical convex body*.

Definition 5 ([4]). Let P be a point of S^{n+1} .

- (1) A spherical convex body \widetilde{W} such that $\widetilde{W} \cap H(-P) = \emptyset$ and $P \in \text{int}(\widetilde{W})$ are satisfied is called a *spherical Wulff shape* relative to P .
- (2) Let \widetilde{W} be a spherical Wulff shape relative to P . Then, the set \widetilde{W}° is called the *spherical dual Wulff shape* of \widetilde{W} .

Definition 6 ([12]). Let \widetilde{W} be a hemispherical subset of S^{n+1} . Then, the following set, denoted by $\text{s-conv}(\widetilde{W})$, is called the *spherical convex hull of \widetilde{W}* .

$$\text{s-conv}(\widetilde{W}) = \left\{ \frac{\sum_{i=1}^k t_i P_i}{\|\sum_{i=1}^k t_i P_i\|} \mid P_i \in \widetilde{W}, \sum_{i=1}^k t_i = 1, t_i \geq 0, k \in \mathbb{N} \right\}.$$

Lemma 2.2 ([12]). *Let $\widetilde{W}_1, \widetilde{W}_2$ be non-empty subsets of S^{n+1} . Suppose that the inclusion $\widetilde{W}_1 \subset \widetilde{W}_2$ holds. Then, the inclusion $\widetilde{W}_2^\circ \subset \widetilde{W}_1^\circ$ holds.*

Proposition 4 ([12]). *For any non-empty closed hemispherical subset $\widetilde{W} \subset S^{n+1}$, the equality $\text{s-conv}(\widetilde{W}) = \left(\text{s-conv}(\widetilde{W}) \right)^\circ$ holds.*

2.3. Construction of Wulff shapes by using spherical polar sets. Let $Id : \mathbb{R}^{n+1} \rightarrow \mathbb{R}^{n+1} \times \{1\} \subset \mathbb{R}^{n+2}$ be the map defined by $Id(x) = (x, 1)$. Denote the north-pole $(0, \dots, 0, 1) \in \mathbb{R}^{n+2}$ by N . The set $S^{n+1} - H(-N)$ is denoted by $S_{N,+}^{n+1}$. Let $\alpha_N : S_{N,+}^{n+1} \rightarrow \mathbb{R}^{n+1} \times \{1\}$ be the central projection relative to N ; namely, α_N is defined as follows where $P = (P_1, \dots, P_{n+1}, P_{n+2}) \in S_{N,+}^{n+1}$ (see Figure 4):

$$\alpha_N(P_1, \dots, P_{n+1}, P_{n+2}) = \left(\frac{P_1}{P_{n+2}}, \dots, \frac{P_{n+1}}{P_{n+2}} \right).$$

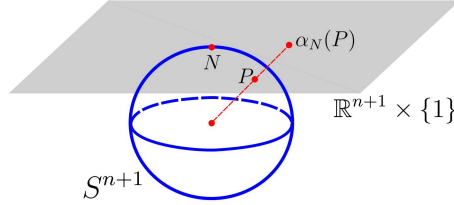


FIGURE 4. The central projection α_N .

Next, we consider the mapping $\Psi_N : S^{n+1} - \{\pm N\} \rightarrow S_{N,+}^{n+1}$ defined by

$$\Psi_N(P) = \frac{1}{\sqrt{1 - (N \cdot P)^2}} (N - (N \cdot P)P).$$

The mapping Ψ_N was firstly introduced in [8]. It has been used for many purposes, for instance for the study of singularities of spherical pedal curves ([8, 9]), for the study of pedal unfoldings of pedal curves ([10]), for the study of hedgehogs ([11]) and for the study of a geometric model of crystal growth in the plane ([6]). The hyperbolic version of Ψ_N is also useful (see [5]). The mapping Ψ_N has the following intriguing properties:

- (1) For any $P \in S^{n+1} - \{\pm N\}$, the equality $P \cdot \Psi_N(P) = 0$ holds.
- (2) For any $P \in S^{n+1} - \{\pm N\}$, the property $\Psi_N(P) \in \mathbb{R}N + \mathbb{R}P$ holds.
- (3) For any $P \in S^{n+1} - \{\pm N\}$, the property $N \cdot \Psi_N(P) > 0$ holds.
- (4) The restriction $\Psi_N|_{S_{N,+}^{n+1} - \{N\}} : S_{N,+}^{n+1} - \{N\} \rightarrow S_{N,+}^{n+1} - \{N\}$ is a C^∞ diffeomorphism.

Moreover, it is easily seen that by using Ψ_N , the inversion $\text{inv} : \mathbb{R}^{n+1} - \{0\} \rightarrow \mathbb{R}^{n+1} - \{0\}$ can be characterized as follows:

Proposition 5.

$$\text{inv} = Id^{-1} \circ \alpha_N \circ \Psi_N \circ \alpha_N^{-1} \circ Id.$$

Proposition 6 ([12]). *Let $\gamma : S^n \rightarrow \mathbb{R}_+$ be a continuous function. Then, \mathcal{W}_γ is characterized as follows:*

$$\mathcal{W}_\gamma = Id^{-1} \circ \alpha_N \left(\left(\Psi_N \circ \alpha_N^{-1} \circ Id(\text{graph}(\gamma)) \right)^\circ \right).$$

Proposition 7 ([12]). *For any Wulff shape \mathcal{W}_γ , the following hold:*

- (1) *The following set, too, is a Wulff shape.*

$$Id^{-1} \circ \alpha_N \left(\left(\alpha_N^{-1} \circ Id(\mathcal{W}_\gamma) \right)^\circ \right).$$

(2) The graph of the convex integrand of \mathcal{W}_γ is as follows.

$$\text{inv} \left(\partial \left(Id^{-1} \circ \alpha_N \left((\alpha_N^{-1} \circ Id(\mathcal{W}_\gamma))^\circ \right) \right) \right).$$

(3) The graph of the convex integrand of $Id^{-1} \circ \alpha_N \left((\alpha_N^{-1} \circ Id(\mathcal{W}_\gamma))^\circ \right)$ is as follows, where $\partial\mathcal{W}_\gamma$ stands for the boundary of \mathcal{W}_γ .

$$\text{inv}(\partial\mathcal{W}_\gamma).$$

The assertions (2), (3) of Proposition 7 has been implicitly proved in [12].

Definition 7 ([12]). For any Wulff shape \mathcal{W}_γ , the Wulff shape

$$Id^{-1} \circ \alpha_N \left((\alpha_N^{-1} \circ Id(\mathcal{W}_\gamma))^\circ \right)$$

given in Proposition 7 is called the *dual Wulff shape* of \mathcal{W}_γ .

3. PROOF OF THEOREM 1

3.1. Proof of the “if” part. In this subsection, we show that W is strictly convex under the assumption that its convex integrand γ_w is of class C^1 . Recall that N (resp., α_N) is the north-pole $(0, \dots, 0, 1)$ of S^{n+1} (resp., the central projection relative to N). Set $\widetilde{W} = \alpha_N^{-1} \circ Id(W)$. By Proposition 6, we have the following:

$$W = Id^{-1} \circ \alpha_N \left((\Psi_N \circ \alpha_N^{-1} \circ Id(\text{graph}(\gamma_w)))^\circ \right).$$

Thus, we have the following:

$$\widetilde{W} = (\Psi_N \circ \alpha_N^{-1} \circ Id(\text{graph}(\gamma_w)))^\circ.$$

By Proposition 5, we have the following:

$$\widetilde{W} = (\alpha_N^{-1} \circ Id \circ \text{inv}(\text{graph}(\gamma_w)))^\circ.$$

By the definition of convex integrand and Proposition 4, the following holds:

$$\partial\widetilde{W}^\circ = \alpha_N^{-1} \circ Id \circ \text{inv}(\text{graph}(\gamma_w)),$$

where $\partial\widetilde{W}^\circ$ stands for the boundary of \widetilde{W}° . Thus, by the assumption of γ_w , the following holds:

Lemma 3.1. *The boundary of \widetilde{W}° is C^1 diffeomorphic to S^n .*

Suppose that W is not strictly convex. Then, since W is convex, there exist two distinct points $x_0, x_1 \in \partial W$ such that

$$\{(1-t)x_0 + tx_1 \mid 0 \leq t \leq 1\}$$

is contained in ∂W . For any $i \in \{0, 1\}$, set $P_i = \alpha_N^{-1}(x_i)$. Then, it follows that

$$P_0 P_1 \subset \partial\widetilde{W}.$$

For any $\tilde{P} \in S^{n+1}$, let $\partial H(\tilde{P})$ be the boundary of $H(\tilde{P})$.

Lemma 3.2. *Under the above situation, the following holds:*

$$\partial H(P_0) \cap \partial H(P_1) \cap \text{int}(H(N)) \subset \widetilde{W}^\circ.$$

Proof. Let P be a point of $\partial H(P_0) \cap \partial H(P_1) \cap \text{int}(H(N))$. Suppose that $P \notin \widetilde{W}^\circ$. Then, there exists a point $Q_0 \in \widetilde{W}$ such that $P \cdot Q_0 < 0$.

On the other hand, we have that $P \cdot P_i = 0$ for any $i \in \{0, 1\}$. Thus, it follows that

$$P \cdot \frac{(1-t)P_0 + tP_1}{\|(1-t)P_0 + tP_1\|} = 0$$

for any $t \in [0, 1]$. Since $P_0P_1 \subset \partial \widetilde{W}$, $P \in \text{int}(H(N))$ and \widetilde{W} is a spherical Wulff shape relative to N , it follows that $P \cdot Q \geq 0$ for any $Q \in \widetilde{W}$.

Thus, we have a contradiction. \square

By Lemma 3.2, for any $P \in \partial H(P_0) \cap \partial H(P_1) \cap \text{int}(H(N))$, we have at least two great hyperspheres $\partial H(P_0), \partial H(P_1)$ which may be candidates for tangent great hyperspheres to \widetilde{W}° at P . This contradicts Lemma 3.1 \square

3.2. Proof of the “only if” part. In this subsection, we show that γ_w is of class C^1 under the assumption that W is strictly convex. We use the same notations given in Subsection 3.1.

Lemma 3.3. *For any point $Q \in \partial \widetilde{W}^\circ$, there exists the unique point $P_Q \in \partial \widetilde{W}$ such that Q is a point of $\partial H(P_Q)$.*

Proof. Suppose that there exists a point $Q \in \partial \widetilde{W}^\circ$ such that $Q \in \partial H(P_0) \cap \partial H(P_1)$, where P_0, P_1 are some distinct points of $\partial \widetilde{W}$. Then, it follows that $P_i \cdot Q = 0$ for any $i \in \{0, 1\}$. This implies that $P \cdot Q = 0$ for any point $P \in P_1P_2$. Then, for any $\varepsilon > 0$ and any point $P \in P_1P_2$, there exists a point \tilde{P} such that two inequalities $\|P - \tilde{P}\| < \varepsilon$ and $\tilde{P} \cdot Q < 0$ are satisfied. Since $Q \in \widetilde{W}^\circ$, it follows that $\tilde{P} \notin \widetilde{W}$ although P belongs to \widetilde{W} . Hence, we have that the arc P_1P_2 is contained in $\partial \widetilde{W}$. This contradicts the assumption that W is strictly convex. \square

Secondly, we show the following lemma:

Lemma 3.4. *The convex integrand $\gamma_w : S^n \rightarrow \mathbb{R}_+$ is differentiable at any $\theta \in S^n$.*

Proof. Let Q be the point of $\partial \widetilde{W}^\circ \subset S^{n+1}$ such that the following is satisfied.

$$\text{inv} \circ Id^{-1} \circ \alpha_N(Q) = (\theta, \gamma_w(\theta)).$$

Let U be a sufficiently small neighbourhood of Q in S^{n+1} . Since $\partial H(P_Q)$ is a great hypersphere and α_N is the central projection relative to N , we may assume that there exists an affine transformation $H : \mathbb{R}^{n+1} \rightarrow \mathbb{R}^{n+1}$ such that $H \circ Id^{-1} \circ \alpha_N(Q)$ is the origin of \mathbb{R}^{n+1} and $H \circ Id^{-1} \circ \alpha_N(\partial H(P_Q) \cap U)$ (denoted by V) is contained in $\mathbb{R}^n \times \{0\}$, where P_Q is the unique point of $\partial \widetilde{W}$ obtained in Lemma 3.3. Let V_1 be a sufficiently small neighborhood of the origin in V . Then, by replacing $H = (h_1, \dots, h_n, h_{n+1})$ with $(h_1, \dots, h_n, -h_{n+1})$ if necessary, we may assume that there exists a continuous function $f_n : V_1 \rightarrow [0, \infty)$ such that $f_n(0, \dots, 0) = 0$ and the graph of f_n is an open subset of $H \circ Id^{-1} \circ \alpha_N(\partial \widetilde{W}^\circ \cap U)$.

We first suppose that $n = 1$. Let a be a positive real number such that $\{x \mid |x| < a\} \subset V_1$. For the a , define $I(a)$ as follows:

$$I(a) = \{\lambda \in \mathbb{R} \mid \exists x \in (-a, 0) \cup (0, a) \text{ such that } f_1(x) = \lambda x\}.$$

The set $I(a)$ has the following properties:

- Claim 3.1.** (1) $0 < a_1 < a_2 \Rightarrow 0 \leq \sup I(a_1) \leq \sup I(a_2)$.
 (2) $a_2 < a_1 < 0 \Rightarrow \inf I(a_2) \leq \inf I(a_1) \leq 0$.
 (3) $\lim_{a \rightarrow 0} \sup I(a) = 0$.
 (4) $\lim_{a \rightarrow 0} \inf I(a) = 0$.

Proof. By definition, (1) and (2) are clear.

We show (3). By (1), the following holds:

$$\lim_{a \rightarrow 0} \sup I(a) \geq 0.$$

Suppose that there exists a positive real number λ_1 such that $\lim_{a \rightarrow 0} \sup I(a) = \lambda_1$. Then, since \widetilde{W}° is spherical convex, the following holds for any x such that $0 < x < a$.

$$f_1(x) \geq \lambda_1 x.$$

Since $\lambda_1 > 0$, the above inequality implies that there exists a point $P \in \widetilde{W}$ ($P \neq P_Q$) such that $Q \in \partial H(P)$. This contradicts Lemma 3.3. Therefore, we have $\lim_{a \rightarrow 0} \sup I(a) = 0$.

(4) may be proved similarly as (3). \square

By Claim 3.1, we have the following

$$\begin{aligned} 0 &\leq \lim_{x \rightarrow +0} \frac{f_1(x)}{x} \leq \lim_{a \rightarrow 0} \sup I(a) = 0, \\ 0 &\geq \lim_{x \rightarrow -0} \frac{f_1(x)}{x} \geq \lim_{a \rightarrow 0} \inf I(a) = 0. \end{aligned}$$

Therefore, γ_w must be differentiable at θ .

Next, we give a proof for general n . Let x be a point of $V_1 - \{0\}$. Set $V_2 = \{0\} \times \mathbb{R} + \mathbb{R}(x, 0) \subset \mathbb{R}^n \times \mathbb{R}$. Then, V_2 is a 2-dimensional real vector space and the intersection

$$\{(x, f_n(x)) \mid x \in V_1\} \cap V_2$$

may be regarded as the graph of f_1 in the case $n = 1$ (see Figure 5). Let

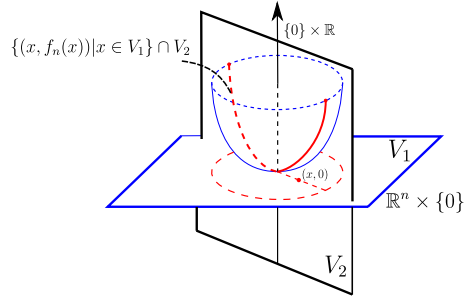


FIGURE 5. $\{(x, f_n(x)) \mid x \in V_1\} \cap V_2$ may be regarded as the graph of f_1 .

$\{r_i\}_{i=1,2,\dots} \subset \mathbb{R}_+$ be a sequence such that $\lim_{i \rightarrow \infty} r_i = 0$. Set $x_i = r_i x$. Then, by the proof in the case $n = 1$, we have the following:

$$\lim_{i \rightarrow \infty} \frac{f_n(x_i)}{\|x_i\|} = 0.$$

Therefore, even for general n , γ_w must be differentiable at θ . \square

Let Q be a point of $\partial \widetilde{W}^\circ$. Let $\{Q_i\}_{i=1,2,\dots} \subset \partial \widetilde{W}^\circ$ be a sequence such that $\lim_{i \rightarrow \infty} Q_i = Q$. By Lemma 3.3, for any Q_i there exists the unique point $P_{Q_i} \in \partial \widetilde{W}$ such that $Q_i \in H(P_{Q_i})$. Set $P = P_Q$ and $P_i = P_{Q_i}$. By Lemma 3.4, in order to show that $\gamma_w : S^n \rightarrow \mathbb{R}_+$ is of class C^1 , it is sufficient to show the following:

$$\lim_{i \rightarrow \infty} h(H(P), H(P_i)) = 0,$$

where $h : \mathcal{H}(S^{n+1}) \times \mathcal{H}(S^{n+1}) \rightarrow \mathbb{R}$ is the Pompeiu-Hausdorff metric. Suppose that there exists a positive real number $\varepsilon > 0$ such that for any $m \in \mathbb{N}$ there exists an integer $i > m$ such that $h(H(P), H(P_i)) > \varepsilon$. Since $Q \in H(P)$ and $Q_i \in H(P_i)$, by the definition of Pompeiu-Hausdorff metric (for the definition of Pompeiu-Hausdorff metric, see for instance [1, 2]), it follows that there exists a positive real number $\varepsilon > 0$ such that for any $m \in \mathbb{N}$ there exists an integer $i > m$ such that $d(Q, Q_i) > \varepsilon$, where $d : \mathbb{R}^{n+2} \times \mathbb{R}^{n+2} \rightarrow \mathbb{R}$ is the Euclidean metric. This contradicts $\lim_{i \rightarrow \infty} Q_i = Q$. Therefore, we have $\lim_{i \rightarrow \infty} h(H(P), H(P_i)) = 0$. \square

4. APPLICATIONS OF THEOREM 1

Since the boundary of the convex hull of a C^1 closed submanifold is a C^1 closed submanifold (for instance, see [14, 19]), as a corollary of Theorem 1, we have the following:

Corollary 1. *Let $\gamma : S^n \rightarrow \mathbb{R}_+$ be a function of class C^1 . Then, \mathcal{W}_γ is strictly convex.*

In particular, we have the following:

Corollary 2 ([12], Theorem 1.3). *Let $\gamma : S^n \rightarrow \mathbb{R}_+$ be a function of class C^1 . Then, \mathcal{W}_γ is never a polytope.*

On the other hand, Figure 6 shows that the converse of Corollary 1 does not hold in general.

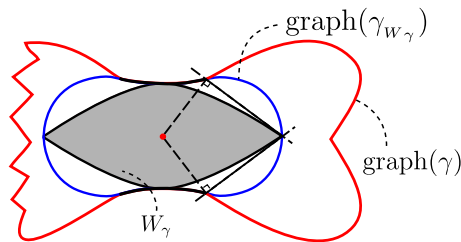


FIGURE 6. A strictly convex \mathcal{W}_γ having non smooth γ .

Combining Theorem 1 and Proposition 7 yields the following:

Corollary 3. *A Wulff shape in \mathbb{R}^{n+1} is strictly convex if and only if the boundary of its dual Wulff shape is C^1 diffeomorphic to S^n .*

In particular, we have the following:

Corollary 4. *A Wulff shape in \mathbb{R}^{n+1} is strictly convex and its boundary is C^1 diffeomorphic to S^n if and only if its dual Wulff shape is strictly convex and the boundary of it is C^1 diffeomorphic to S^n .*

It is interesting to compare Corollary 4 and the following:

Proposition 8 ([12]). *A Wulff shape in \mathbb{R}^{n+1} is a polytope if and only if its dual Wulff shape is a polytope.*

Finally, we give an application of Theorem 1 from the view point of pedal.

Definition 8. Let p (resp., $F : S^n \rightarrow \mathbb{R}^{n+1}$) be a point of \mathbb{R}^{n+1} (resp., a C^1 embedding). Then, the *pedal of $F(S^n)$ relative to p* is the mapping $G : S^n \rightarrow \mathbb{R}^{n+1}$ which maps $\theta \in S^n$ to the nearest point in the tangent hyperplane to $F(S^n)$ at $F(\theta)$ from the given point p .

Let W be a Wulff shape in \mathbb{R}^{n+1} . Suppose that ∂W is C^1 diffeomorphic to S^n . Then, ∂W may be regarded as the graph of a certain C^1 embedding $F : S^n \rightarrow \mathbb{R}^{n+1}$, and γ_W is exactly the pedal of ∂W relative to the origin. Theorem 1 gives a sufficient condition for the pedal of ∂W relative to the origin to be smooth:

Corollary 5. *Suppose that a Wulff shape W in \mathbb{R}^{n+1} is strictly convex and its boundary is C^1 diffeomorphic to S^n . Then, the pedal of ∂W relative to the origin is of class C^1 .*

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